

Chapter 1

The transverse-field Ising chain

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1.1 Introduction

The Ising model in a transverse field is the simplest quantum many-body system that has a quantum phase transition. It consists of a chain of spin- $\frac{1}{2}$ particles which is coupled by a ferromagnetic exchange interaction J and which are placed in a magnetic field h that points perpendicular to the easy axis. As is well known, the competition between the two terms leads to a transition between an ordered and a disordered phase at zero temperature. In thermodynamic limit the transition happens at a critical value of the field, and the behaviour of the chain close to this point is universal.

It is worth noting that it is the case that the model can be solved exactly. This was first done by Pfeuty in 1976 using a mapping to free fermions, which had been introduced earlier by Jordan and Wigner [1, 2]. Because of the exact solution, the transverse-field Ising chain is the standard example against which numerical methods are tested [3]. In this chapter we receive the exact solution and compare it with numerics for finite chains.

The chapter is organised as follows. Section 1.2 introduces the model and the notation. In Section 1.3 the Jordan–Wigner transformation is applied and the dispersion of the fermionic excitations is derived. Section 1.4 discusses the phase diagram, which is shown in Figure ??, and Section 1.5 compares the exact ground-state energy with exact diagonalisation of short chains.

1.2 Model and notation

We consider a chain of L sites with periodic boundary conditions. On every site j there is a spin- $\frac{1}{2}$ degree of freedom, described by the Pauli matrices σ_j^x , σ_j^y and σ_j^z . The Hamiltonian reads

$$H = -J \sum_{j=1}^L \sigma_j^z \sigma_{j+1}^z - h \sum_{j=1}^L \sigma_j^x, \quad (1.1)$$

with $\sigma_{L+1}^z \equiv \sigma_1^z$. The first term favours states in which neighbouring spins is aligned along the z axis; the second term favours states in which every spin points along x . Throughout this chapter we take $J > 0$ and $h \geq 0$.

Two limits are easy to understand. For $h = 0$ the Hamiltonian is diagonal in the Z basis, and the two ground states are the fully polarised states $|\uparrow\uparrow \cdots \uparrow\rangle$ and $|\downarrow\downarrow \cdots \downarrow\rangle$. This is the ferromagnetic phase, in which the $\mathbb{Z}_2$ symmetry generated by $P = \prod_j \sigma_j^x$ is spontaneously broken. For $h \rightarrow \infty$ the ground state is the unique product state in which every spin is polarised

along z , and the symmetry is unbroken. This is the paramagnetic phase. Between the two limits there has to be a phase transition, and it is the aim of this chapter to locate it and to analyse its properties.

We use the order parameter

$$m_z = \frac{1}{N} \sum_{j=1}^L \langle \sigma_j^z \rangle, \quad (1.2)$$

which is nonzero in the ferromagnetic phase and vanishes in the paramagnetic phase. In the paramagnetic phase this quantity is not an unique measure of the disorder, so we will additionally use the energy gap Δ between the ground state and the first excited state.

1.3 Jordan–Wigner transformation

The key idea of the exact solution is to represent the spins by spinless fermions. We define

$$c_j = \left(\prod_{l < j} \sigma_l^x \right) \sigma_j^-, \quad \sigma_j^\pm = \frac{1}{2} (\sigma_j^z \pm i \sigma_j^y), \quad (1.3)$$

where the string of σ^x operators ensures that the c_j satisfy the canonical anticommutation relations $\{c_j, c_l^\dagger\} = \delta_{jl}$. Inserting Equation (1.3) into Equation (1.1) gives a quadratic form in the fermions,

$$H = -J \sum_j (c_j^\dagger - c_j)(c_{j+1}^\dagger + c_{j+1}) - h \sum_j (1 - 2c_j^\dagger c_j), \quad (1.4)$$

up to a boundary term that depends on the fermion parity and that we ignore for sufficiently large chains. Since the Hamiltonian is quadratic it can be diagonalised by a Fourier transformation followed by a Bogoliubov transformation. The result is

$$H = \sum_k \varepsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right), \quad \varepsilon_k = 2\sqrt{J^2 + h^2 - 2Jh \cos k}, \quad (1.5)$$

where the γ_k are fermionic quasiparticle operators and k runs over the allowed momenta in the first Brillouin zone. The dispersion in Equation (1.3) is shown in the left panel of Figure ?? for three values of the field.

The energy gap is the minimum of the dispersion, which is reached at $k = 0$:

$$\Delta = \varepsilon_0 = 2|J - h|. \quad (1.6)$$

Hence the gap closes exactly at $h = J$, and this is the location of the quantum phase transition. Note that the gap $\Delta = 4|J - h|$ is linear in the distance from the critical point on both sides. The ground-state energy follows from Equation (1.5) as

$$E_0 = -\frac{1}{2} \sum_k \varepsilon_k \xrightarrow{L \rightarrow \infty} -\frac{L}{2\pi} \int_{-\pi}^{\pi} \sqrt{J^2 + h^2 - 2Jh \cos k} dk. \quad (1.7)$$

At the critical point $h = J$ the integral can be done in closed form and gives $E_0/L = -4J/\pi$.

1.4 The phase diagram

The two phases identified in Section 1.2 are separated by a single critical point at $h_c = J$, see Fig. 1.1. In the ordered phase, $h < h_c$, the order parameter m_z is finite and the gap is $\Delta = 2(J - h)$. In the disordered phase, $h > h_c$, the order parameter vanishes and the gap is $\Delta = 2(h - J)$. Exactly at the critical point the gap closes and the correlation length diverges

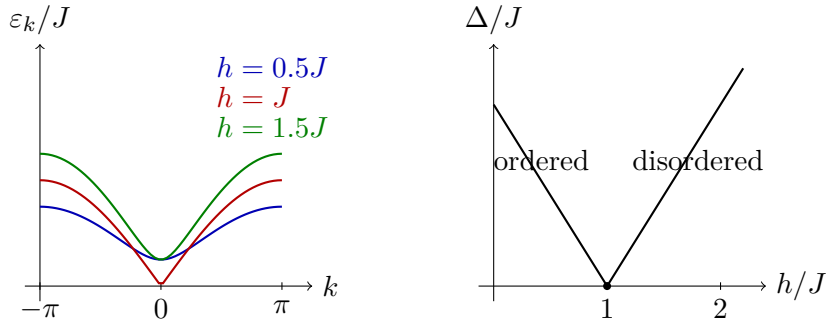


Figure 1.1: Left: the quasiparticle dispersion ε_k of Equation (1.5) for three values of the transverse field. Right: the gap $\Delta = 2|J - h|$ as a function of the field; it closes at the critical point $h_c = J$.

Table 1.1: Ground-state energy per site at $h = J$ for periodic chains of length L from exact diagonalisation, compared with the thermodynamic limit $-4/\pi$ of Equation (1.7). The values are given in units of J .

L	E_0/L	deviation
8	-1.2872	1.1×10^{-2}
12	-1.2795	4.9×10^{-3}
16	-1.2769	2.9×10^{-3}
20	-1.2757	1.9×10^{-3}
∞	-1.2732	—

as $\xi \sim |h - h_c|^{-1/2}$, that is, with the critical exponent $\nu = 1/2$ of the two-dimensional classical Ising model.

This mapping between the quantum chain and the classical two-dimensional model is not a coincidence. A bunch of arguments show that the transfer matrix of the classical model in the limit of an anisotropic lattice is the exponential of the quantum Hamiltonian, so that the universal properties of both are identical. This is the reason why the transverse-field Ising chain was shown to have been found to be the prototype of a quantum critical point.

The transversal field also has a simple interpretation in terms of the fermions: it is a chemical potential. In the disordered phase the fermionic band is empty, and the excitations are single fermions that cost at least Δ . In the ordered phase the picture is similar, but the elementary excitations are domain walls between the two ferromagnetic ground states. It is remarkable that these two very different pictures are both described by the same dispersion (1.5).

1.5 Numerical check

To analyze the accuracy of exact diagonalisation for short chains, we compare the ground-state energy per site of chains with up to $L = 20$ sites with the exact result of Equation (1.7). The Hamiltonian (1.1) is represented as a sparse matrix of dimension 2^L and its lowest eigenvalue is obtained with the Lanczos algorithm. Table 1.1 lists the results at the critical point.

The finite-size corrections decrease as $1/L^2$, as expected for a critical point with dynamical exponent $z = 1$, where the leading correction is set by the central charge of the conformal field theory. Fitting the four data points to $E_0/L = e_\infty + a/L^2$ gives $e_\infty = -1.2733$, in agreement with the exact value to four digits. This confirms that the implementation is correct and that it can be used for the perturbed models of the following chapters, for which no exact solution exists.

1.6 Summary

The transverse-field Ising chain can be mapped to free fermions with the dispersion of Equation (1.5). The gap closes at $h_c = J$, which separates a ferromagnetic phase from a paramagnetic phase. Its critical exponents are those of the two-dimensional classical Ising model. Exact diagonalisation reproduces the exact ground-state energy with an error below 10^{-2} already for $L = 8$ sites, and the extrapolation in $1/L^2$ agrees with the exact value to four digits.

Bibliography

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